Astronomy for GCMs

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1 The Earth's Orbit

We can assume that the Earth's orbit is an ellipse with eccentricity ϵ. The eccentricity varies on geologic ($\approx 100,000$ year) time scales. At the current time, it is rather small, $\epsilon = 0.0167$; 120,000 years ago it was ≈ 0.04. For a plot of the eccentricity versus time, see Held (1982). The definition of an ellipse and its eccentricity is such that the equation for the distance from one focus (the sun) to the orbit is

$$\frac{r(\omega)}{a} = \frac{1-\epsilon^2}{1+\epsilon\cos(\omega-\lambda)} \tag{1}$$

where a is the semi-major axis and ω is an angle subtended by the Earth in its orbit, with $\omega = \lambda$ the perihelion of the orbit. Our convention will be that $\omega = 0$ at autumnal equinox inthe Northern Hemisphere. iven this convention, we will refer to λ as the longitude of the perihelion. Currently $\lambda = 102.9°$, so that perihelion occurs in late January. The longitude of the perihelion passes through 2π over $\approx 20,000$ years.

According to Newtonian mechanics, the period of a planet in orbit around the sun is a function only of the semi-major axis a. Furthermore, a has no "secular" variation on long time scales do to perturbations by the other planets, unlike the eccentricity, so a and the length of a "year" can be taken as fixed. The area of the ellipse is

$$A = \pi a^2(1-\epsilon^2)^{1/2} \tag{2}$$

We now define τ to be the time normalized so that one year $= 2\pi$, with $\tau = 0$ once again at autumnal equinox. We need to be able to compute the position of the Earth on its orbit as a function of time, so as to generate the Earth-Sun distance as needed. Since equal areas are subtended in equal times (one of Kepler's laws) one has

$$dA = r^2 d\omega/2 = Cd\tau. \tag{3}$$

with C a constant. Integrating over the year, the left-hand side reduces to the area of the ellipse, so that

$$C = \frac{A}{2\pi} = \frac{a^2}{2}(1-\epsilon^2)^{1/2} \tag{4}$$

and

$$\frac{d\omega}{d\tau} = \frac{a^2}{r^2}(1-\epsilon^2)^{1/2} \tag{5}$$

or

$$\frac{d\omega}{d\tau} = \frac{1+\epsilon\cos(\omega-\lambda)}{(1-\epsilon^2)^{3/2}} \tag{6}$$

While one can approximate the solution to this equation for small ϵ it is easy to integrate it numerically to obtain $\omega(\tau)$ for any ϵ. In the astronomy module this numerical integration is performed with a 4'th order Runge-Kuttta scheme, so as to generate ω at a series of standard values of τ. One then interpolates on this table to get the value at any parficular value of τ.

2 Zenith Angle and solar flux

See Sellers, 1965 for a discussion of some of the following expressions.

The Earth's axis of rotation makes an angle Φ with the normal to the orbital plane, an angle referred to as the *obliquity*. Currently, $\Phi = 23.4°$. The obliquity varies with roughly a 40,000 year period between 22.5° and 25°. (See Held (1982) for a plot of obliquity versus time.)

Now consider a line from the sun to the center of the earth, intersecting the earth's surface at the *sub-solar point*. The latitude of this point is the *declination*, δ. At the (summer,winter) solstice, $\omega = (3\pi/2, \pi/2)$, we have $\delta = (+\Phi, -\Phi)$, while at the equinoxes, $\omega = (0, \pi)$, we have $\delta = 0$. (Note: summer and winter always refer to the Northern Hemisphere.) More generally,

$$\sin(\delta) = -\sin(\Phi)\sin(\omega) \tag{7}$$

To prove this, consider a coordinate system with its z-axis normal to the orbital plane, and with the x-axis pointing towards the point on the orbit of autumnal equinox. Then a unit vector in the direction of the rotation axis is $\hat{\Omega} = (0, sin(\Phi), cos(\Phi))$. A unit vector pointing from the sun to the Earth will be $\hat{r} = (\cos(\omega), sin(\omega), 0)$. With this construction, $\hat{r}\cdot\hat{\Omega} = cos(\pi/2-\delta) = -sin(\delta)$ which yields the desired result. (A figure would be useful here)

When one is located at the subsolar point, the sun is directly overhead. At any other point on the Earth, there is a non-zero angle between the local vertical and the a line from the center of the Earth to the Sun. This angle is the *zenith* angle, Z. The zenith angle is most conveniently computed as a function of latitude ϕ, longitude, h, and the declination of the sun δ. The longitude h is measured with respect to the longitude of the subsolar point, and is referred to a the *hour angle.* At local noon, $h = 0$. Now place a Cartesian coordinate system so that the the $x - y$ plane lies in the Earth's equatorial plane, with the x-axis oriented so that the subsolar point lies in the $x - z$ plane. Then a unit vector pointing towards the sun from the center of the Earth is $\hat{S} = (\cos(\delta), 0, \sin(\delta))$ while a unit vector pointing towards local vertical is $\hat{V} = (\cos(\phi)\cos(h), \cos(\phi)\sin(h), \sin(\phi))$. Since $\cos(Z) = \hat{S} \cdot \hat{V}$, we obtain the classic result that

$$\cos(Z) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(h) \tag{8}$$

(A figure would be useful here as well)

The length of the "day", during which the sun is above the horizon ($Z > 0$) can be obtained by setting $Z = 0$ and solving for h, which will then be the *half-day length*, denoted by H.

$$\cos(H) = -\tan(\phi)\tan(\delta) \tag{9}$$

The half-day length H equals $\pi/2$ (6 hours) at the equator. It also equals $\pi/2$ at all latitudes at the time of the equinoxes ($\delta = 0$).

Setting $H = 0$, we can compute the latitude boundary of polar night, $\tan(\phi_{pn}) = -\cot(\delta)$. Polar day occurs whenever $H > \pi$, or $\tan(\phi_{pd}) = \cot(\delta)$

We can average $\cos(Z)$ over the daylight hours:

$$\cos(Z_d) = (\sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\sin(H)/H) \tag{10}$$

Here Z_d is the zenith angle that would produce the correct insolation if the Sun were held fixed at that zenith angle during the daylight hours.

The incident solar flux, per unit area perpendicular to the solar beam is $Q = S_0(r/a)^2$, where S_0 is the *solar constant*, defined here as the incident solar flux at the distance from the sun equal to the semi-major axis. One often sees the solar constant defined as the flux at a distance equal to the "mean Earth-Sun distance", but I think that this difference is much smaller than the observed ($\approx 0.1\%$) variations in S_0 observed to occur in the 11-year solar cycle, so is not worth worrying about (check!). (The expression "mean Earth-Sun distance" is somewhat ambiguous anyway – is this a time mean or a mean averaged over the orbit?)

If the distance from the sun does not vary significantly over the length of a day, and the declination does not vary significantly as well, then the insolation averaged over the entire day is

$$Q_d = S_0(\frac{r}{a})^2 \cos(Z_d)H/\pi \tag{11}$$

since H/π is the fraction of the day in which there is light. In regions of polar night, $Q = 0$ of course, but in regions of polar day one should be careful to set $H = \pi$. (This expression will be inaccurate for planets with length of day that is not short compared with length of the year.)

In diurnally averaged models, one often simply fixes the zenith angle at Z_d and sets the insolation equal to Q_d. It would presumably be more accurate to average $cos(Z)$ over the day, weighted by Q, or, equivalently, averaging $cos^2(Z)$.

3 Siderial versus Solar days

The siderial day is 2π divided by the angular velocity of the Earth's rotation (with respect to the rest of the universe).

The solar day is the time between noon on one day and noon on the next. It is 86400 seconds by the definition of the second (more or less). If the siderial day were equal to one year, the Earth would rotate so that the same point was always facing the sun, and the solar day would be infinite. More generally, one must subtract the angular velocity of the planet's motion around the sun from its rotational angular velocity to obtain the length of the solar day

$$\frac{2\pi}{solarday} = \frac{2\pi}{siderialday} - \frac{2\pi}{year} \tag{12}$$

REFERENCES

Held, I. M., 1982: Climate Models and the Astronomical Theory of the Ice Ages, *Tellus*, **50**, 449-461.

Sellers, W. D., 1965: *Physical Climatology*, The University of Chicago Press, Chicago, 272pp..